3D Interactive Plots for Multivariate Calculus

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Accessibility Statement

Accessibility features of the web version of this resource

The web version of 3D Interactive Plots for Multivariate Calculus has been designed with accessibility in mind by incorporating the following features to ensure it has been optimized for users of screen reader technology:

- All content can be navigated using a keyboard
- Links, headings and tables are formatted to work with screen readers
- Images have text descriptions
- Information is not conveyed by colour alone
- The option to increase the font size (see tab on the top right of screen)

Other file formats available

This Pressbook is only available in a web version due to the interactive GeoGebra graphs that are included in each of the learning units.

Known accessibility issues and areas for improvement

While we strive to ensure that this resource is as accessible as possible, we might not always get it right. There may be some supplementary third-party materials, or content not created by the authors of this book, which are not fully accessible. This may include videos that do not have closed captioning or accurate closed captioning, inaccessible PDFs, etc. Any issues we identify will be listed below. There are currently no known issues.

Some pages feature an interactive plotting applet called GeoGebra. For more information on how to use a keyboard or screen reader to navigate applets, please view GeoGebra's accessibility page.

Accessibility standards

The web version of this resource has been designed to meet Web Content Accessibility Guidelines 2.0, level AA. In addition, it follows all guidelines in Accessibility Toolkit: Checklist for Accessibility. The development of this toolkit involved working with students with various print disabilities who provided their personal perspectives and helped test the content.

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Introduction

About This Resource

The Purpose of This Resource

In this book, you will find a pool of interactive and colorful three-dimensional (3D) graphs with supplemental self-checking questions. The topics covered in this book have been selected to improve both teaching and learning vital concepts and techniques in multivariable calculus, one of the fundamental courses across the undergraduate curriculum in science and engineering.

The 3D graphs in this resource were developed using an open-source graphing tool (Geogebra). The units in this resource have been organized based on the most used open-source textbook in this subject area, Calculus Volume 3 by OpenStax3, to ensure both learners and instructors have free access to a high-quality open education resource (OER) in this area that is accessible and inclusive by design.

This resource was designed to apply learner-centered design principles, aiming to (a) engage diverse learners and develop their geometric intuition about abstract and complex mathematical concepts (e.g., partial derivatives, multiple integrals, vector fields) and (b) train learners to make connections between concepts visually (e.g., connecting "vectors" in mathematics with "magnitude" and "direction" in physics) and thereby prepare them well to understand more fully engineering, physics and mathematical problems (e.g., differential equations) in their subsequent STEM coursework.

What to Expect

Within each unit, you'll find the following sections:

- **The Concept:** In this section, we'll share the key concepts you'll need to know for the unit topic.
- **The Plot:** In this section, we'll provide step-by-step instructions on engaging with a 3D plot related to the unit topic. Following the instructions, you should be able to manipulate the 3D graph to understand the key concepts for the unit.
- **Self-Checking Questions:** In this section, you'll find questions to test your understanding of the unit concepts. Answers to some of the questions will be provided; however, some questions will only be able to be found through using the 3D plot in the unit.

Using This Resource

What You Need

To go through the units in this resource, you will need:

- Access to a computer and the Internet
- Basic knowledge of the derivatives and integrals of single-variable functions
- Around 15-20 minutes is needed per unit, on average

How to Use This Resource

This resource was created for undergraduate students across Ontario.

For learners using this resource, you will most likely be assigned a specific topic or unit by your instructor as part of your undergraduate coursework.

For instructors using this resource, you may link directly to the resource as a whole or by the unit in your teaching. This resource

How to Navigate the Modules/Units

This resource is meant to be used for learning at your own pace and according to your own needs, i.e., you can go through the units, including the plots and self-checking questions as many times as you find relevant to your academic needs.

The resource is hosted in Pressbooks (a web-based platform). If you are unfamiliar with Pressbooks, please view the video below to learn how to navigate Pressbooks [1].

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://pressbooks.library.ryerson.ca/multivariatecalculus/?p=578#oembed-1

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Copyright Statement

Units

Unit 1: Partial Derivatives

The Concept

When studying derivatives of functions of one variable $y=f(x)$, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is $\frac{dy}{dx}$, which implies that y is the dependent variable and x is the independent variable. $\frac{dy}{dx}$ also represents the slope of the tangent line at a certain point of this function.

For a function $z=f(x,y)$ of two variables, x and y are the independent variables (input to function f) and z is the dependent variable (output of function z, the value of z is depend on the values of x and y). We will have two partial derivatives and their Leibniz notations are $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$. They are analogous to ordinary derivatives:

$$\frac{\partial z}{\partial x}(x_0, y_0)=\frac{\text{change in } z}{\text{change in }x}(\text{holding }y \text{ as a constant } y_0)$$

$$\frac{\partial z}{\partial y}(x_0, y_0)=\frac{\text{change in } z}{\text{change in }y}(\text{holding }x \text{ as a constant } x_0)$$

Besides the Leibniz notations above, you can also write the derivatives as $\frac{\partial z}{\partial x}=f_x$ and $\frac{\partial z}{\partial y}=f_y$. Similar to the geometric meaning of $\frac{dy}{dx}$ in two-dimensional (2D), $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in three-dimensional (3D) represent the slopes of tangent lines as well.

The Plot

Now, you should engage with the 3D plot below for partial derivatives[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Input a function of two variables, then set y as a constant, e.g., -1. A cross-section plane $y=-1$ is plotted. Recall that the function $y = y_0$ (or $x=x_0$) in 3D represents the planes that are perpendicular to the *xy*-plane.
2. A tangent line passing through the point (x_0, y_0) and also on the cross-section plane $y=y_0$ is plotted.

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Change the y-values using the slider, and you will see the cross-section and the tangent line changes. You can also rotate the graph to get a better view. Since the particle derivative is the slope of the tangent line, the partial derivative $\frac{\partial z}{\partial x}$ changes as well.

3. Repeat the same steps in (1) and (2) for $\frac{\partial z}{\partial y}$.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. Let $f(x,y)=\frac{xy}{x-y}$. Find $f_x(2,-2)$ and $f_y(2,-2)$.
2. The apparent temperature index, A,

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is a measure of how the temperature feels,

$$A=0.885x -22.4 y +1.2 xy -0.544$$

where x is relative humidity and y is the air temperature. Find $\frac{\partial A}{\partial x}$ and $\frac{\partial A}{\partial y}$ when $x=20°F$ and $y=1$.

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Unit 2: Tangent Plane

The Concept

For a 2D curve $y=f(x)$, there is at most one **tangent line** to a point (x_0, y_0) on the curve. The **equation of tangent line** to 2D curve $y=f(x)$ at point (x_0, y_0) is

$$y=y_0+f'(x_0)(x-x_0).$$

The **tangent plane** in 3D is an extension of the above tangent line in 2D. For a 3D surface $z=f(x,y)$, there are infinitely many tangent lines to a point (x_0, y_0, z_0) on the surface; these tangent lines lie in the same plane and they form the tangent plane at that point.

Recall that two lines determine a plane in 3D space. Thus, one usually uses two special tangent lines to determine a tangent plane and these two tangent lines are related to the partial derivatives (i.e., f_x and f_y) of the surface function $z = f(x,y)$. The **equation of the tangent plane** to surface $z = f(x,y)$ at point (x_0, y_0, z_0) is z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y-y_0).

The Plot

Now, you should engage with the 3D plot below to understand the tangent plane[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Input a 3D surface function in the function box in the plot. The function can be a single variable function or a double variable function.
2. Adjust point P using the sliders or by dragging the point on the graph below.
3. The tangent plane equation will be depicted on the plot.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. Find the equation of the tangent plane to the surface defined by the function $x^2+10xyz+y^2+8z^2=0, P(-1,-1,-1)$
2. Find the equation of the tangent plane to the surface defined by the function $h(x,y) = \ln(x^2) + y^2$ at Point $(x_0,y_0) = (3,4)$.

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Unit 3: Directional Derivative

The Concept

Directional derivatives look to extend the concept of **partial derivatives** by finding the tangent line parallel to neither the x-axis or y-axis.

We start with the graph of a surface defined by the equation $z = f(x,y)$. Given a point (x_0, y_0) in the domain of $f(x,y)$, we choose a direction defined by a **unit vector** \vec{u}=\langle a,\,b\rangle, where $a^2+b^2=1$, to travel from that point. This **direction vector** can also be written as $\vec{u}= \langle \cos\theta,\,\sin\theta \rangle$. Angle θ is measured counterclockwise on the xy-plane, starting at zero from the positive x-axis. The derivative along that direction (that is, the directional derivative) represents the traveling speed and it is defined as the dot product between the **gradient vector**, $\nabla f = \langle f_x, f_y \rangle$, and direction vector, $\vec{u}$, i.e.,

$$D_u f(x_0,y_0) = \nabla f \cdot \vec{u}=f_x(x_0, y_0) \, a+ f_y(x_0, y_0) \,b$$

,

where $a=\cos\theta$ and $b=\sin \theta$.

Consider two special cases of directional derivatives:

1. When $\theta = 0$, we travel in the direction that is parallel to positive x-axis, so the direction $\vec{u} = \langle \cos0, \, \sin0\rangle = \langle 1,\,0\rangle$ and the corresponding directional derivative is $D_u f(x,y) = f_x(x_0,y_0)\,1+ f_y(x_0,y_0)\,0= f_x(x_0,y_0)$.
2. When $\theta = \frac{\pi}{2}$, we travel in the direction that is parallel to positive y-axis, so the direction $\vec{u} = \langle \cos \frac{\pi}{2} , \, \sin \frac{\pi}{2} \rangle = \langle 0,\, 1 \rangle$ and the directional derivative is $D_u f(x_0,y_0) = f_x(x_0,y_0)\,0+ f_y(x_0,y_0)\,1= f_y(x_0,y_0)$.

The concept of directional derivatives can be extended into high dimensions. For example, we consider the 3D gradient vector, $\nabla f= \langle f_x, f_y, f_z \rangle$ and 3D direction vector, $\vec{u}=\langle a,\,b\, c \rangle$, where $a^2+b^2+c^2=1$ because

$\vec{u}$ is a unit vector. Thus the directional derivative of $w = f(x,y,z)$ at point (x_0, y_0, z_0) is

$$D_u f(x_0,y_0, z_0) = \nabla f \cdot \vec{u}=f_x(x_0, y_0, z_0)\, a + f_y(x_0, y_0, z_0) \,b + f_y(x_0, y_0, z_0) \,c$$

.

The Plot

Now, you should engage with the 3D plot below to understand directional derivatives[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. There are two separate plots where the direction vector (i.e., the direction of the derivative, denoted by $\vec{u}$) is defined by either an angle in radians or a vector.
2. Point (P) is adjusted with the x and y sliders.
3. $\vec{u}$ is selected by either the angle or vector and is indicated by the red arrow on the graph.

Directional Derivative (Defined by an angle)

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Direction Derivative (Defined by a vector)

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. Find the gradient, $\nabla f(x,y)$, of the function: f(x,y) = x^2 - xy + 3y^2
2. Find the directional derivative, D_u f, of the function: f(x,y,z) = e^{-2z} sin(2x)cos(2y) at point (0,1).

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Unit 4: Local Extrema and Saddle Points

The Concept

We use the **first derivative test** and **second derivative test** to locate and distinguish between local minima, local maxima and saddle points for a function $z = f(x,y)$.

First derivative test

For functions of a single variable, $y = f(x)$, **critical points** in 2D are defined as the values of the function in which the derivative $\frac{df}{dx}$ equals zero or which does not exist. When dealing with functions of two variables, $z = f(x,y)$, the concept of critical points in 3D remains virtually identical, save for the fact that we must now deal with partial derivatives. Thus, for functions of two variables, $z = f(x,y)$, in order to find the critical points (x_0, y_0), we need to solve a system of two equations:

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

Second derivative test

Similarly, the most important quantity in the second derivative test is **the Jacobian matrix**, denoted as 'J'. It is the matrix of all its second-order partial derivatives, i.e.,

$$J=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}=\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Note that $f_{xy}=f_{yx}$. We plug in the critical points from the first derivative into the Jacobian and calculate **the determinant of the Jacobian matrix**, denoted as 'D', i.e.,

$$D=\begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}=f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Then we use the following rules to conduct the second derivative test:

1. If $D>0$ and $f_{xx}(x_0, y_0)>0$, then f has a

local minimum at (x_0, y_0).

2. If D>0 and f_{xx}(x_0, y_0)<0, then f has a **local maximum** at (x_0, y_0).
3. If D<0, then f has a **saddle point** at (x_0, y_0).
4. If D=0, then the test is inconclusive.

The Plot

Now, you should engage with the 3D plot below to understand local maximum, local minimum and saddle points[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Input a function of two variables into the f(x,y) input function section.
2. Move the point on the plane around and the Jacobian determinant will automatically be calculated for you. The equation for each is provided, where the determinant of the jacobian represents the D value from the formula above.
3. Once you hover over a local maximum, local minimum or a saddle point, a text will appear notifying you of the answer.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

Use the first and second derivative tests to identify any critical points and determine whether each critical point is a maximum, minimum, saddle point or none of these.

1. f(x,y) = -x^3+4xy-2y^2+1
2. f(x,y) = 2xye^{-x2-y2}

Use the graph to find the answers to these questions.

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Unit 5: Double Integral Over Rectangular Regions

The Concept

The definition of the **single integral** in 2D space is as follows: given a **single-variable function** $y=f(x)$ that is continuous on the interval $[a,b]$, we divide the interval into n subintervals of equal width, x, and from each interval choose a point, x_i. **Definite integral,** $\int_a^bf(x)dx$, represents the area inbetween the curve, $y=f(x)$, and x-axis. **Riemann sum** helps to approximate such areas, that is,

$$\int_a^bf(x)dx \approx \sum_{i=1}^n f(x_i) \Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i=a + i\Delta x$. The larger n is, the better the estimation is. Thus, the limit of the Riemann sum defines the definite integral,

$$\int_a^bf(x)dx =\lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^{\infty} f(x_i)\,\Delta x$$

.

Similar to the single integral, the **double integral** in 3D, $\iint_R f(x,y) dA$, is equal to the volume under the surface of the **two-variable function** $z = f(x,y)$ and above the region R on the xy-plane. Here, we consider this region has a very simple shape, rectangle, and use R to denote it. The x coordinate of this rectangle changes from a to b, and y coordinate changes from c to d, denoted as $R=[a, b]\times[c,d]$. As in the case of the single integral, a **double integral** is defined as the limit of a **Reimann sum**, i.e.,

$$\iint_R f(x,y) dA=\lim_{m,n \to \infty}\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i},y_{j}) \Delta A$$

where
$\Delta A=\Delta x \Delta y, \Delta x=\frac{b-a}{n}, \Delta y=\frac{d-c}{m}, x_i=a + i\Delta x$
and $y_i=c + i\Delta y$.

The Plot

Now, you should engage with the 3D plot below to understand double integrals over rectangular regions[1]. Follow the steps below to apply changes to the plot and observe the effects:

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1. Input a function of two variables into the $f(x,y)$ input function section.
2. Move the n-slide around to decide the subregions of the rectangular region, R, and we consider the subregions are squares.
3. Pick xmin, xmax, ymin, and ymax points for your domain/bounds of the rectangular region, R.
4. Use the k-slider to choose which square-shaped subregion you'd like to highlight.
5. Use the checkboxes to show either all of the rectangular prisms compared to just the one you are highlighting, as well as whether to see the graph or not.
6. By changing your view and hovering over the plot, you can see a 2D representation of the rectangular area. Additionally, the double integral is dynamically calculated at the bottom.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

Calculate the integrals by interchanging the order of integration:

1. $\int_{-1}^{1} \int_{-1}^{2} 2x + 3y + 5 \, dx dy$
2. $\int_{0}^{\pi} \int_{0}^{\pi/2}sin(2x) cos(3y)\, dx dy$

Use the graph to find the answers to these questions.

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Unit 6: Double Integrals Over the General Region

The Concept

Now, you should engage with the 3D plot below to understand the double integral over the general region (i.e., non-rectangular region). There are two types of double integrals.

- **Type I double integral**: $\int_{a}^{b} \int_{h(x)}^{g(x)} f(x,y)\,dy dx$, where $x=a$ and $x=b$ are the lower and upper bounds of x; $y=h(x)$ and $y=g(x)$ are the lower and upper bounds of y.
- **Type II double integral:** $\int_{a}^{b} \int_{h(x)}^{g(x)} f(x,y)\,dx dy$, where $y=a$ and $y=b$ are the lower and upper bounds of y; $x=h(y)$ and $x=g(y)$are the lower and upper bounds of y.

You may notice that the bounds of outer integrals (a and b) for both Type I and Type II integrals are constants; these two integrals are “symmetric” – if you switch x and y in Type I, you get Type II and vice versa.

The Plot

Now, you should engage with the plot below to understand double integrals with general regions[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Assume you have a Type I integral $\int_{0}^{1} \int_{-x}^{x^2} y^2 x\,dy dx$. Input $y^2 x$ into the $f(x,y)$ input function section.
2. Input $\textrm{If} (0 \leq x \leq 1, x^2)$ into the Upper y function, i.e., $g(x)$ section.
3. Input $\textrm{If} (0 \leq x \leq 1, -x)$ into the lower y function, i.e., $h(x)$ section.
4. Use the slider for the value of x to see the change of the area of the cross-section, $A(x)$.
5. The result of this double integral is dynamically calculated at the bottom.
6. You can also use this plot for the Type II integral by switching x and y.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. $\int_{0}^{1} \int_{2\sqrt{x}}^{2\sqrt{x}+1} xy+1\,dy dx$
.
2. $\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2x+4x^3\,dx dy$

Use the graph to find the answers to these questions.

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Unit 7: Double Integrals in Polar Coordinates

The Concept

Now we will look at the concept of **double integrals in polar coordinates**. Rather than using a **cartesian (or rectangular) coordinate system** as we have used thus far to evaluate single and double integrals, we will use the polar coordinate system. The **polar coordinate system** is a 2D coordinate system in which each point on a plane is determined using a distance from a reference point and an angle from a reference direction. The rectangular coordinate system is best suited for graphs and regions that are naturally considered over a rectangular grid. The polar coordinate system is an alternative that offers good options for functions and domains that have more circular characteristics.

While a point P in rectangular coordinates is described by an ordered pair (x,y), it may also be described in polar coordinates by (r, θ), where r is the distance from P to the origin and θ is the angle formed by the line segment and the positive xx-axis. We may convert a point from rectangular to polar coordinates using the following equations:

$$r =\sqrt{x^2+y^2} \text{ and } \tan(\theta) = \frac{y}{x},$$

or convert a point from polar to rectangular coordinates using the following equations:

$$x =r \cos\theta \text{ and } y = r \sin\theta.$$

The double integral $\iint_D f(x,y)\,dA$ in rectangular coordinates can be converted to a double integral in polar coordinates as $\iint_D f(r \cos\theta, r \sin\theta)\,r\,dr d\theta$.

The Plot

Now, you should engage with the plot below to understand polar coordinates[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Change the bounds on the double integral in polar coordinates for both the r and θ bounds. The bounded region will be shown in the plot and t in the plot represents θ.
2. The result of the double integral in polar coordinates will be shown too.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. \iint_{D} 3x \, dA where R={(r,\theta)| 0 \leq r \leq 1, 0 \leq \theta \leq 2}
.
2. \iint_{D} 1-x^2-y^2 \, dA where R={(r,\theta)| 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi}

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.

Use the graph to find the answers to these questions.

Unit 8: Triple Integral in Rectangular Coordinate

The Concept

The definition of the **double integral** was introduced in Unit 5. Just as we use a double integral to integrate over a 2D region, we use a **triple integral**, $\iiint_D f(x,y,z)\,dV$, to integrate over a 3D region. Similarly, as with double integrals, the bounds of inner integrals may be functions of the outer variables. These bound functions are what encode the shape of the general region D. We may define a triple integral generally as follows:

$$\iiint_D f(x,y,z)\,dV=\int_{a}^{b} \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz dy dx$$

where $x=a$ and $x=b$ represent the lower and upper bounds of x, $y=g1(x)$ and $y=u2(x,y)$ are the lower and upper bounds of y, and $z=u1(x,y)$ and $z=u2(x,y)$ are the lower and upper bounds of z. Similar to double integrals, triple integrals are iterative as well. Thus, they can be written as different forms, such as

$$\iiint_D f(x,y,z)\,dV=\int_{c}^{d} \int_{g_1(y)}^{g_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \, dz dx dy$$

.

The Plot

Now, you should engage with the 3D plot below to understand triple integrals in rectangular coordinates[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. You are able to change the bounds on the triple integral in rectangular coordinates.
2. You may input your function, $f(x,y,z)$, to be integrated at the bottom as well, in which the triple integral of said function will be presented at the top of the screen in the beige area.
3. You may also change the grid size of the 3D solid depicted on the screen for the function $f(x,y,z) = 1$.

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Self-Checking Questions

Check your understanding by solving the following questions[2].

Evaluate the triple integrals over the rectangular solid box B.

1. $\iiint_D (2x + 3y^2 + 4z^3)\, dV$, where
   $B=\{(x,y,z)| 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$
2. $\iiint_D z \sin(x) + y2) \, dV$, where
   $B=\{(x,y,z)| 0 \leq x \leq \pi, 0 \leq y \leq 12, -1 \leq z \leq 2\}$

Use the graph to find the answers to these questions.

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Unit 9: Triple Integrals in Cylindrical Coordinates

The Concept

Cylindrical coordinates are a simple extension of 2D **polar coordinates** to 3D. Recall that, in Unit 7, the position of a point in 2D (i.e., xy-plane) can be described using polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the angle between the x-axis and the line segment from the origin to the point. With the addition of a third dimension, z-axis from the Cartesian (i.e., rectangular) coordinate system, we are able to describe a point in 3D cylindrical coordinates, i.e., (r, θ, z).

Cylindrical coordinates simply combine the polar coordinates in the xy-plane with the usual z coordinate of Cartesian coordinates. To form the cylindrical coordinates of a point P, simply project it down to a point Q in the xy-plane. Then, take the polar coordinates (r, θ) of the point Q. The third cylindrical coordinate is the same as the usual z-coordinate. It is the signed distance of point P to the xy-plane.

The Plot

Now, you should engage with the 3D plot below to understand triple integrals in cylindrical coordinates[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Change the bounds on the triple integral in cylindrical coordinates where and represent the outermost bounds. \alpha and \beta are constants and they are the lower and upper bounds of angle \theta. r_1 and r_2 are functions of \theta), and they are the lower and upper bounds of r. u_1 and u_2 are functions of r and \theta, and they are the lower and upper bounds of z.
2. You may also change the grid size of the 3D solid depicted on the screen for the function f(x,y,z).

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

Evaluate the triple integrals $f(x,y,z)$ over the solid E.

1. $E = \{(x,y,z) | x^2+y^2 \leq 9, x \geq 0, y \geq 0, 0 \leq z \leq 1\}$, $f(x,y,z) = z$
2. $E = \{(x,y,z) | 1 \leq x^2+y^2 \leq 9, y \geq 0, 0 \leq z \leq 1\}$, $f(x,y,z) = x^2+y^2$

Use the graph to find the answers to these questions.

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Unit 10: 3D Solid Bounded by Two Surfaces

The Concept

The graphs of **functions of two variables** $z=f(x, y)$ are examples of surfaces in 3D. More generally, a set of points (x,y,z) that satisfy an equation relating all three variables is often a surface. A simple example is the unit sphere, the set of points that satisfy the equation $x^2+y^2+z^2=1$.

One special class of equations is a set of equations that involve one or more x^2, y^2, z^2, xy, xz, and yz. The graphs of these equations are surfaces known as **quadric surfaces**. There are six different quadric surfaces: the ellipsoid, the elliptic paraboloid, the hyperbolic paraboloid, the double cone and hyperboloids of one sheet and two sheets. Quadric surfaces are natural 3D extensions of the so-called conics (ellipses, parabolas and hyperbolas), and they provide examples of fairly nice surfaces to use as examples in multivariate calculus.

The Plot

Now, you should engage with the 3D plot below to understand 3D solids bounded by two surfaces[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Fill in function 1 (i.e., f(x,y,z)) and function 2 (i.e., g(x,y,z)) with your desired quadric surfaces.
2. The graph depicted on the right shows their intersection.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

Plot the given quadric surface and specify the name of said quadric surface:

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1. $x^2/4 + y^2/9 - z^2/12 = 1$
2. $z^2 = 4x^2 + 3y^2$

Use the graph to find the answers to these questions.

Unit 11 : Vector Fields in 2D and 3D

The Concept

A **vector field** is an assignment of a vector to each point in a subset of space. In other words, if we are given a vector $\langle x,y\rangle$, then the vector is simply the mapping in 2D of each point.

Vector fields can be written in two equivalent notations shown below for both 2D and 3D:

- **2D Notation:**
  $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} = \langle P(x, y) , Q(x, y)\rangle$
- **3D Notation:**
  $\vec{F}(x,y,z) = P(x, y,z)\vec{i}+ Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} = \langle P(x, y,z) , Q(x, y,z) , R(x,y,z)\rangle$

Where $\vec{i} = \langle 1 , 0 \rangle$ and $\vec{j} = \langle 0,1\rangle$ represent **unit vectors** in 2D, and $\vec{i} = \langle1,0,0\rangle ,\, \vec{j} = \langle 0,1,0 \rangle$ and $\vec{k} = \langle 0,0,1\rangle$ are unit vectors in 3D. A real life example that can be modeled as a vector field would be a fluid dynamics problem such as a river, where the velocity of the liquid is a vector at any given point. The magnitude (i.e., amplitude) of the vector represents the speed and the direction represents the direction of the flow at any given point.

The Plot

Now, you should engage with the 2D and 3D plots below to understand 2D and 3D vector fields[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. The vector definition is done using P and Q.
2. Y Grid and X Grid control the number of arrows that will appear in the 2D plot.
3. Xmin and Ymin set the minimum boundaries for the plot.
4. YMax and Xmax set the maximum boundaries for the plot.

2D Vector Field Plot

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3D Vector Field Plot

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. Draw the following vector field \vec{F}(x,y)=x\vec{i} + y\vec{j}
2. Draw the following vector field \vec{F}(x, y, z) = 2x\vec{i} − 2y\vec{j} − 2z\vec{k}

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Use the graphs to find the answers to these questions.

Unit 12: Line Integrals

The Concept

For a **single-variable integral** in 2D, $\int_a^b f(x)\,dx$, we integrate function $f(x)$ along x in 2D and it represents the area inbetween the curve, $y=f(x)$, and a segment of x-axis from a to b .

A **line integral** in 3D shares a similar idea to a single-variable integral in 2D. A **line integral**, $\int_C f(x,y)\,ds$, integrates the surface function, $z=f(x,y)$, along a 2D curve segment C on the xy-plane, instead of x on the x-axis or y on the y-axis alone. This line segment, C, is described by a vector function, $r(t)=\langle x(t), y(t) \rangle$, where $t=a$ and $t=b$ map the start point and end point of C, respectively. The differential element, ds, represents the change of arc length of curve C, i.e., $ds=\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \,\,dt$. Thus the line integral can be evaluated by the following single integral:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \,\,dt$$

.

The value of $\int_C f(x,y)\,ds$ is the area of the “wall”, “fence” or “curtain” whose base is the 2D curve C on the xy-plane and and whose height is given by the function $z=f(x,y)$.

The concept of line integral can be extended to high dimensions. For example, $\int_C f(x,y,z)\,ds$ integrates the function with three variables, $w=f(x,y,z)$, along a 3D curve C that is parameterized by $r(t) = \langle x(t),y(t),z(t) \rangle$. It can be evaluated by a single integral as well, that is,

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2+(\frac{dz}{dt})^2} \,\,dt$$

.

The Plot

Now, you should engage with the 3D plot below to understand tangent planes [1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Input the function $f(x, y)$.
2. Adjust the 2D curve C on the xy-plane.
3. Adjust the number of rectangular subareas, n.

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4. The estimation of the line integral is shown. The larger n is, the better the estimation is.

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Self-Checking Questions

Check your understanding by solving the following questions[2]:

1. Find the value of integral \int_C(x^2+y^2)\, ds, where C is part of the helix parameterized by r(t)=\langle cos t, sin t \rangle, 0 \leq t \leq 2
.
2. Evaluate \int_C \frac{1}{x^2+y^2} \, ds , over the line segment from $(1,1)$ to $(3,0)$.

2. Gilbert Strang, Edwin "Jed" Herman, OpenStax, Calculus Volume 3,

Use the graph to find the answers to these questions.

Unit 13: Work

The Concept

Work is the measurement of a force on an object along a line through a 2D or 3D vector field. An intuitive example would be kayaking upstream in a river. The river or water source would be the vector field $\vec{F}$ and the object would be the kayak, the path would be the route we take upstream with the kayak, and, lastly, the work is effort used to overcome the current. Understand that the current will have different magnitudes and directions at different points–this is why we represent this with a vector field.

Mathematically, the definition of a vector field $\vec{F}$ in 2D or 3D is given as

$$\vec{F}(x,y) = \langle P(x, y) , Q(x, y) \rangle \text{ or}$$
$$\vec{F}(x,y,z)= \langle P(x, y,z) , Q(x, y,z) , R(x,y,z)\rangle$$

Looking to answer the question of how we can compute the work done by the river of moving the kayak along route C, we can calculate the work W done by force field $\vec{F}$ along the curve C as the following equation

$$W = \int_C \vec{F} \cdot dr= \int_a^b F(r(t)) r'(t)\,dt$$

The Plot

Now, you should engage with the 2D plot below to understand work[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Fill in $P(x, y)$ and $Q(x, y)$ (i.e., the first and second compartments of the vector field function).
2. Adjust xmin, xmax, ymin and max and they are the lower and upper bounds for the x-axis and y-axis.
3. Input the function for curve C $y=f(x)$ (i.e., the trajectory that the object travels along).
4. Adjust a and b (i.e., the x-coordinates of the start and end points of curve C).
5. The result of the work is shown.

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Self-Checking Questions

Check your understanding by solving the following question[2]:

1. Find the work done by vector field $\vec{F}(x,y)=y\vec{i}+2x\vec{j}$ in moving an object along path C, which joins points (1,0) and (0,1).

Use the graph to find the answer to this question.

2. Gilbert Strang, Edwin “Jed” Herman, OpenStax, Calculus Volume 3,

Unit 14: Flux in 3D

The Concept

Let us introduce the idea of **flux** with a typical application. We are given a **vector field** $\vec{F}= \langle P(x, y,z) , Q(x, y,z) , R(x,y,z) \rangle$ that represents the flow of a fluid, for example $\vec{F}$ represents the velocity of wind in 3D. The flux is the rate of the flow per unit time. The flux of $\vec{F}$ across surface S is the line integral denoted by $\int_{S} \vec{F} \cdot n(t)\, ds$, where $\vec{F}$ is a vector field, surface S is defined by $g(x,y,z) = 0$, and $\vec{n}=\frac{\nabla g}{||\nabla g||}$ is represents the unit normal vector and $\nabla g=\langle \frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z} \rangle$. Imagine surface S is a membrane across which fluid flows, but S does not impede the flow of the fluid. In other words, S is an idealized membrane invisible to the fluid. Suppose F represents the velocity field of the fluid.

The Plot

Now, you should engage with the 3D plot below to understand flux[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Fill in P(x, y,z), Q(x, y,z) and R(x, y,z)(i.e., three compartments of the vector field function).
2. Input the surface function.
3. The graph depicted shows the flux.

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Self-Checking Questions

Check your understanding by solving the following question[2]:

1. Consider the radial field

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$\vec{F}(x,y,z)= \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)}$ and sphere S centred at the origin with radius 1. Find the total outward flux across S.

Use the graph to find the answer to this question.

Unit 15: Divergence and Curl

The Concept

Divergence of vector field $\vec{F}$ is defined as an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{F}$ at a particular point P in 2D or 3D is a scalar measure of the "outflowing-ness" of the vector field $\vec{F}$ at point P.

Mathematically, we can define divergence as:

- If $\vec{F}=\langle P(x,y), Q(x,y) \rangle$ is a vector field in 2D, and P_x and Q_y both exist, then the divergence of $\vec{F}$ is defined by
  $div(F) =P_x+Q_y = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$
  .
- If $\vec{F}=\langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a vector field in 3D and P_x, Q_y, and R_z all exist, then the divergence of $\vec{F}$ is defined by

$$div(F) = P_x+Q_y+R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

.

In other words, $div(F)$ is equal to the rate of change of P, Q or R in each respective direction added together. If we think of this logically and add the rate of change in each direction at a specific point, then we will get the rate of change and direction at that point.

Curl of vector field $\vec{F}$ is denoted as $curl(\vec{F})$, which measures the extent of rotation of the field about a point. Suppose that $\vec{F}$ represents the velocity field of a fluid. Then, the curl of $\vec{F}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector.

- If $\vec{F}=\langle P, Q \rangle$ is a vector field in 2D, and P_x and Q_y both exist, then the curl of $\vec{F}$ is defined by
  $$Curl(\vec{F}) = (Q_x - P_y)\vec{k} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$$
  .
- If $\vec{F}=\langle P, Q ,R\rangle$ is a vector field in 3D, and P_x, Q_y, and R_z all exist, then the curl of $\vec{F}$ is defined by

Curl(\vec{F}) = (R_y-Q_z)\vec{i} + (P_z-R_x)\vec{j} + (Q_x - P_y)\vec{k} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \rangle

Divergence and curl are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the fundamental theorem of calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is **conservative** if and only if its curl is zero.

The Plot

Now, you should engage with the plot below to understand divergence and curl[1]. Follow the steps below to apply changes to the plot and observe the effects:

1. Fill in a field function.
2. Choose a path.
3. The graph depicted shows the divergence and curl.

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Self-Checking Questions

Check your understanding by solving the following question[2]:

1. Find the divergence and Curl of D_u f of the function: f(x,y,z) = x(cos(y))\vec{i} +xy^2\vec{j}

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2. Gilbert Strang, Edwin "Jed" Herman, OpenStax, Calculus Volume 3,